CSE 3400 - Introduction to Computer & Network Security
(aka: Introduction to Cybersecurity)

Lecture 10
Public Key Cryptography– Part I

Ghada Almashaqbeh
UConn

From Textbook Slides by Prof. Amir Herzberg
UConn

Outline

- Number theory review.
- Intro to public key cryptography.
- Key exchange.
- Hardness assumptions: DL, CDH, DDH.

Number Theory Review

Our Focus

- A brief overview of mainly modular arithmetic.
  - The minimalist set we need in topics covered in this course.

The Modulo Operation

Definition 1.2 (The modulo operation)**.** *Let* $a, m \in \mathbb{Z}$ *be integers such that* $m > 0$. *We say that an integer* r *is a* residue *of* a *modulo* m *if* $0 \leq r < m$ *and* $(\exists\, i \in \mathbb{Z})(a = r + i \cdot m)$. *For any given* $a, m \in \mathbb{Z}$, *there is exactly one such residue of* a *modulo* m*; we denote it by* $a \mod m$.

Properties (make it easier to compute complex modular arithmetic expressions):

$$(a+b) \mod m = [(a \mod m) + (b \mod m)] \mod m \tag{1.2}$$

$$(a-b) \mod m = [(a \mod m) - (b \mod m)] \mod m \tag{1.3}$$

$$a \cdot b \mod m = [(a \mod m) \cdot (b \mod m)] \mod m \tag{1.4}$$

$$a^b \mod m = (a \mod m)^b \mod m \tag{1.5}$$

The Modulo Operation

Properties (extends also to polynomials):

Similar properties hold for any polynomial $p(x)$ with integer coefficients and input $(x \in \mathbb{Z})$, as well as for a polynomial $p(x_1, x_2, \ldots)$ with integer coefficients and multiple integer parameters $(x_1, x_2, \ldots \in \mathbb{Z})$:

$$\begin{aligned} [p(x)] \mod m &= [p(x \mod m)] \mod m && (1.6) \\ [p(x_1,\ x_2, \ldots)] \mod m &= p(x_1 \mod m, x_2, \ldots) \mod m = && (1.7) \\ &= p(x_1 \mod m, \ldots) \mod m && (1.8) \end{aligned}$$

Examples

- 7 mod 9 = ?
- 13 mod 8 = ?
- 0 mod 11 = ?
- 4 mod 4 = ?
- (30 + 66) mod 11 = ?
- How about: $445 \cdot \left(81 \cdot 34^{13} + 83 \cdot 33^{345}\right) \mod 4$?

Denote $445 \cdot \left(81 \cdot 34^{13} + 83 \cdot 33^{345}\right) \mod 4$ by x. Then we find x as follows:

$$
\begin{aligned}
x &= 445 \cdot \left(81 \cdot 34^{13} + 83 \cdot 33^{345}\right) \mod 4 \\
&= (445 \mod 4) \cdot \big((81 \mod 4) \cdot (34 \mod 4)^{13} + \\
&\quad +(83 \mod 4) \cdot (33 \mod 4)^{345}\big) \mod 4 \\
&= 1 \cdot \left(1 \cdot 2^{13} + 3 \cdot 1^{345}\right) \mod 4 \\
&= \left(2 \cdot 4^{6} + 3\right) \mod 4 \\
&= 3 \mod 4 = 3
\end{aligned}
$$

Multiplicative Inverse

- Needed to support division in modular arithmetic.
  - Division not always produce integers.
  - Modular arithmetic requires integers to work with!!
- To compute ***a/b mod m***, multiply ***a*** by the multiplicative inverse of ***b***.
  - That is compute $a/b \bmod m = ab^{-1} \bmod m$.
  - Where b^{-1} is the multiplicative inverse such that $bb^{-1} \bmod m = 1$
- Not all integers have multiplicative inverses with respect to a specific modulus m.

Multiplicative Inverse

Definition *Let $x, m \in \mathbb{Z}$ be integers such that x and m are a coprime.* . *Then there is a unique integer x^{-1} such that $x \cdot x^{-1} \mod m = 1$ and $m > x^{-1} > 0$. We say that x^{-1} is the* multiplicative inverse *of x modulo m.*

- Examples:
  - 3/5 mod 4 = 3 . 5^{-1} mod 4 = ?
  - 3/5 mod 6 = 3 . 5^{-1} mod 6 = ?

- The algorithm used to compute the inverse is called the Extended Euclidean algorithm (out of scope for this course).

Modular Exponentiation

- Will be encountered a lot; discrete log-based scheme, RSA, etc.
- We have seen a property to reduce the base, but how about the exponent?
  - Its reduction will be with respect to a different modulus than the one in the original operation.
- Fermat's Little Theorem:

Theorem 1.1. *For any integers $a, b, p \in \mathbb{Z}$, if p is a prime and $p > 0$, then*

$$\begin{aligned} a^b \mod p &= a^{b \mod (p-1)} \mod p \\ &= (a \mod p)^{b \mod (p-1)} \mod p \end{aligned} \tag{1.9}$$

Modular Exponentiation

- Examples; Use Fermat's Little theorem (if applicable) to solve the following:
- $13^{32} \bmod 31 = ?$
- $19^{930} \bmod 4 = ?$
- $19^{60} \bmod 7 = ?$

- Can we reduce the exponent for non-prime (composite) modulus?
  - We can use Euler's Theorem.

Euler's Function

- Called also Euler's Totient function. For every integer n ≥ 1, this function computes the number of positive integers that are less than or equal to n and co-prime to n.

$$\phi(n) = |\{i \in \mathbb{N}: i \leq n \land \gcd(i,n) = 1\}|$$

Examples:

n	1	2	3	4	5	6	7	8	9	10
$\phi(n)$	1	1	2	2	4	2	6	4	6	4
factors?	none	none	none	$2 \cdot 2$	none	$2 \cdot 3$	none	2^3	$3 \cdot 3$	$2 \cdot 5$

Euler's Function Properties

Lemma 1.1. *For any prime* $p > 1$ *holds* $\phi(p) = p - 1$*. For prime* $q > 1$ *s.t.* $q \neq p$ *holds* $\phi(p \cdot q) = (p-1)(q-1)$*.*

Lemma 1.2 (Euler function multiplicative property)**.** *If* a *and* b *are co-prime positive integers, then* $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$*.*

Lemma 1.3. *For any prime* p *and integer* $l > 0$ *holds* $\phi(p^l) = p^l - p^{l-1}$*.*

Lemma 1.4. *Let* $n = \Pi_{i=i}^{n}\left(p_i^{l_i}\right)$*, where* $\{p_i\}$ *is a set of distinct primes (all different), and* l_i *is a set of positive integers (exponents of the different primes). Then:*

$$\phi(n) = \phi\left(\Pi_{i=i}^{n}\left(p_i^{l_i}\right)\right) = \Pi_{i=1}^{n}\left(p_i^{l_i} - p_i^{l_i-1}\right) \tag{1.12}$$

Euler's Theorem

Theorem 1.2 (Euler's theorem)**.** *For any co-prime integers* m, n *holds* $m^{\phi(n)} = 1 \mod n$. *Furthermore, for any integer* l *holds:*

$$m^l \mod n = m^{l \mod \phi(n)} \mod n \qquad (1.19)$$

- Examples:
  - 13^{31} mod 31 = ?
  - 27^{26} mod 10 = ?

Last Stop

- Congruence: $a \equiv b \pmod{m}$
  - Used when two expressions have the same residue with respect to some modulus.
  - It is an equivalence relation, so it satisfies:

Reflexivity: $a \equiv a \pmod{m}$.

Symmetry: $a \equiv b \pmod{m}$ if $b \equiv a \pmod{m}$.

Transitivity: if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$.

- Lastly, we have the fundamental theorem of arithmetic.

Theorem 1.3 (The fundamental theorem of arithmetic)**.** *Every number $n > 1$ has a unique representation as a product of powers of distinct primes.*

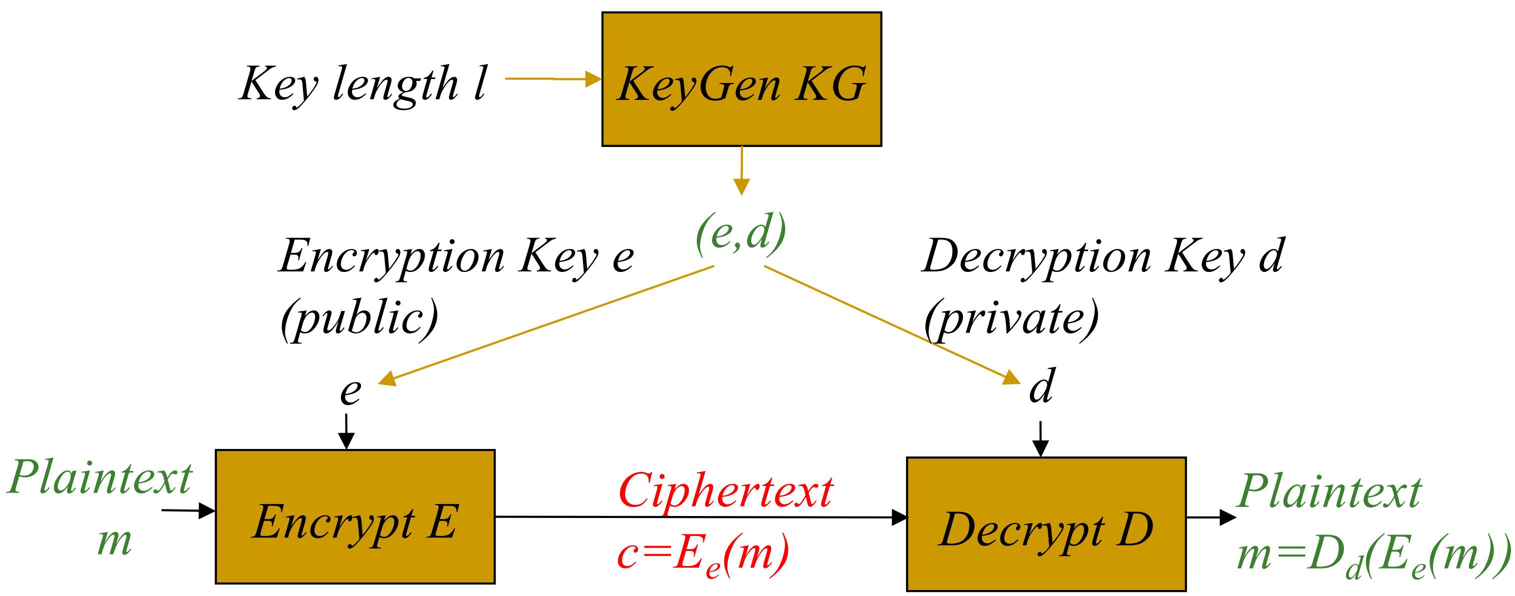
Intro to Public Key Cryptography

Public Key Cryptology

- Kerckhoff: cryptosystem (algorithm) is public
- What we learned until now:
  - Only the key is secret (unknown to attacker)
  - Same key for encryption, decryption
    ➔ if you can encrypt, you can also decrypt!
- But can we give encryption capability without a decryption capability?
  - Yes, using public key cryptography!

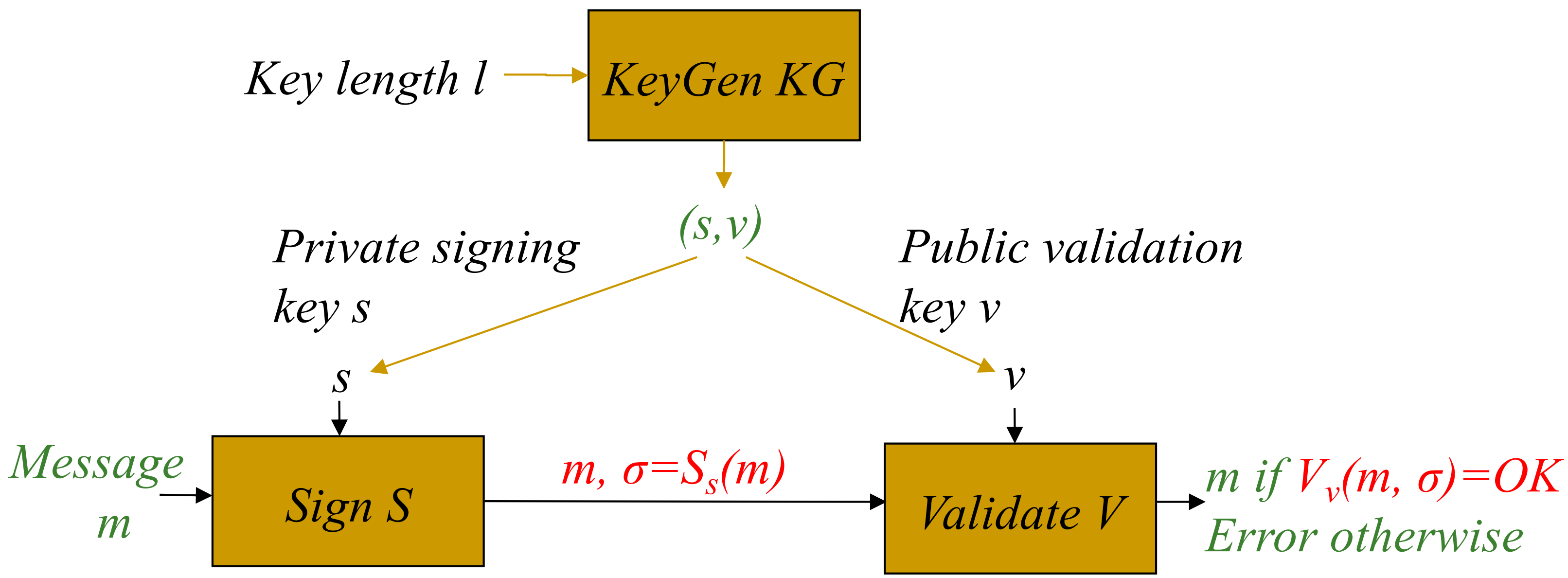
Public Key Cryptosystem (PKC)

- Kerckhoff: cryptosystem (algorithm) is public
- [DH76]: can encryption key be public, too??
  - Decryption key will be different (and private)
  - Everybody can send me mail, only I can read it.



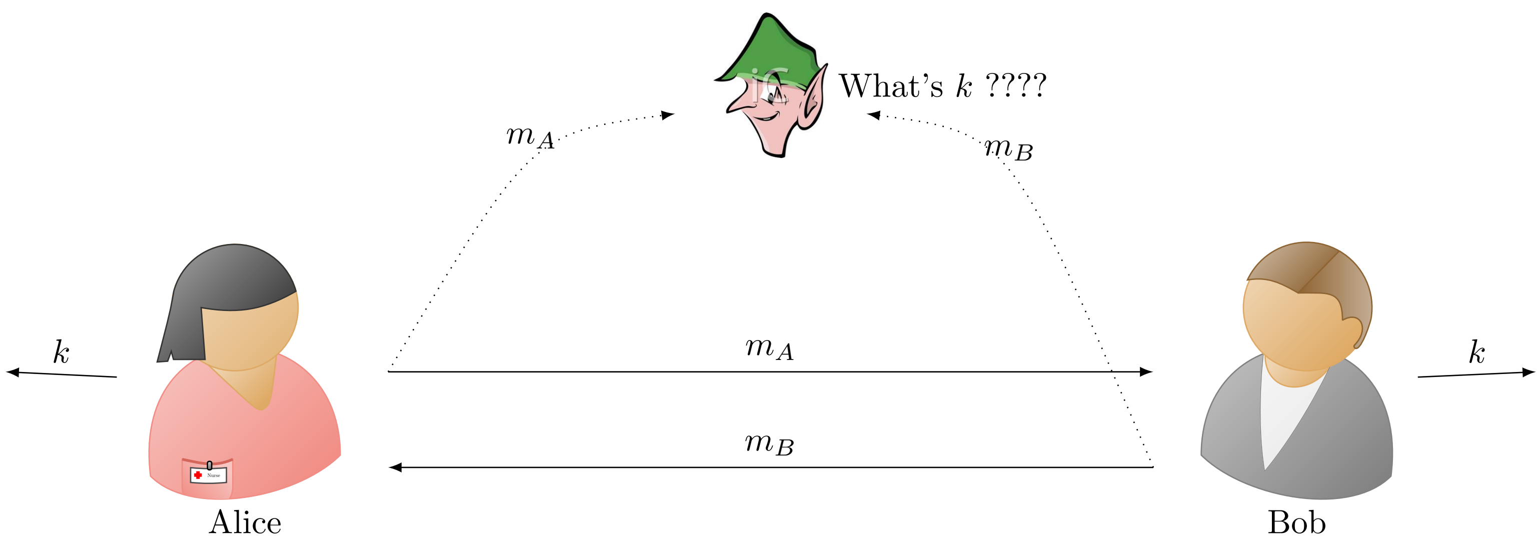
Is it Only About Encryption?

- Also: Digital signatures
  - Sign with private key s, verify with public key v
  - (Recall MACs; a shared key cryptosystem for message authentication).



More: Key-Exchange Protocol

- <u>Key Exchange Protocols</u>
  - Establish shared key between Alice and Bob **without** assuming an existing shared ('master') key !!
  - Use public information from A and B to setup shared secret key k.
  - Eavesdropper cannot learn the key k.



Public keys solve more problems…

- Signatures provide **evidences**
  - Everyone can validate, only ‘owner’ can sign
- Establish shared secret keys
  - Use authenticated public keys
    - Signed by trusted certificate authority (CA)
  - Or: use DH (Diffie Hellman) key exchange
- Stronger resiliency to key exposure
  - Perfect forward secrecy and recover security
    - Protect confidentiality from possible key exposures
  - Threshold (and proactive) security
    - Resilient to exposure of k out of n parties (every period)

Public keys are easier…

- To distribute:
  - From directory or from incoming message (still need to be authenticated)
  - Less keys to distribute (same public key to all)
- To maintain:
  - Can keep in non-secure storage as long as being validated (e.g. using MAC) before using
  - Less keys: $O(|parties|)$, not $O(|parties|^2)$
- So: why not **always** use public key crypto?

The Price of PKC

- Assumptions
  - Applied PKC algorithms are based on a small number of specific computational assumptions
    - Mainly: hardness of factoring and discrete-log
  - Both may fail against quantum computers
- Overhead
  - Computational
  - Key length
  - Output length (ciphertext/signature)

Public key crypto is harder…

- Requires related public, private keys
  - Private key `reverses` public key
  - Public key does not expose private key
- Substantial overhead
  - Successful cryptanalytic shortcuts → need long keys
  - Elliptic Curves (EC) may allow shorter key (almost no shortcuts found)
  - Complex computations
  - RSA: very complex (slow) key generation
- Most: based on hard modular math problems

[LV02]	Required key size		
Year	AES	RSA, DH	EC
2010	78	1369	160
2020	86	1881	161
2030	93	2493	176
2040	101	3214	191

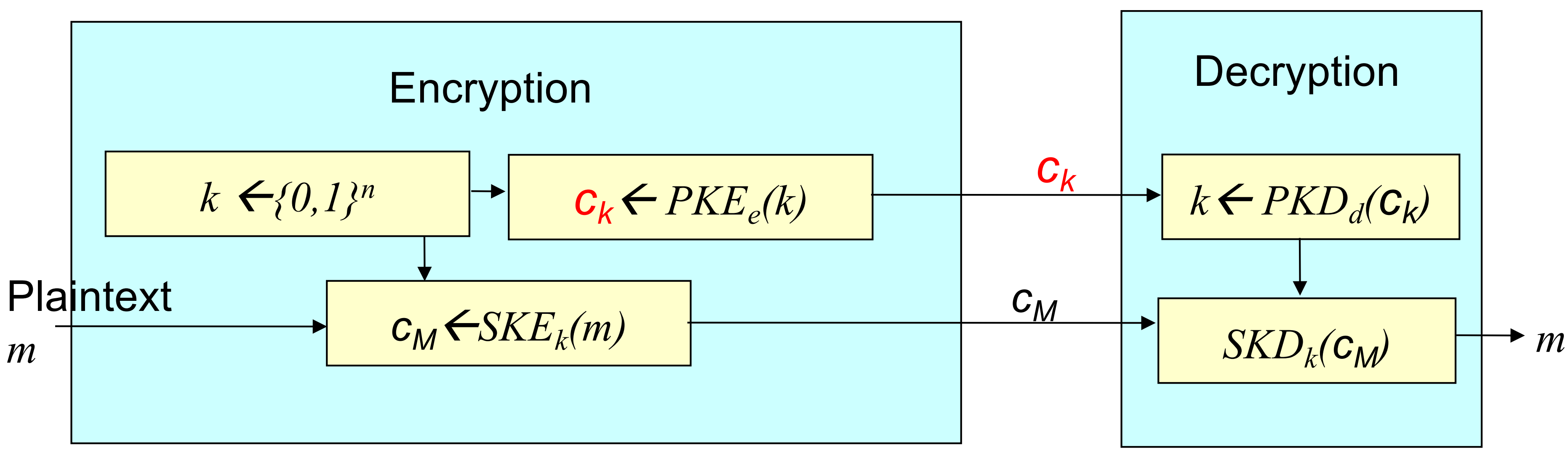
Commercial-grade security
Lenstra & Verheul [LV02]

In Sum

- Minimize the use of PKC
- In particular: apply PKC only to **short inputs**
- How ??
  - For signatures:
    - **Hash-then-sign**
  - For public-key encryption:
    - **Hybrid encryption**

Hybrid Encryption

- Challenge: public key cryptosystems are slow
- Hybrid encryption:
  - Use a shared key encryption scheme to encrypt all messages.
  - But use a public key encryption system to exchange the shared key (Alice generates the k, encrypt it under Bob's public key and send it to Bob, Bob can then recover this key).



Hard Modular Math Problems

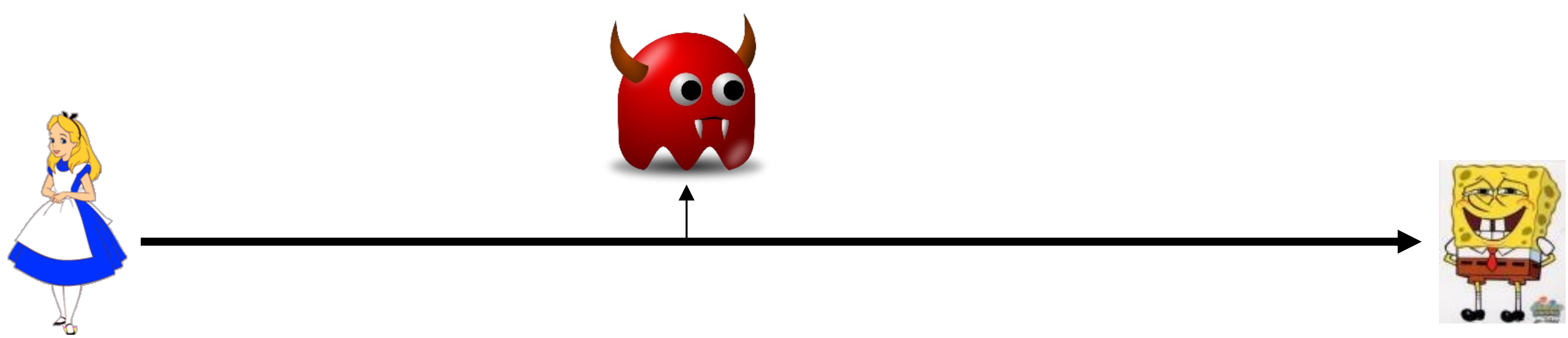
- No efficient solution, in spite of extensive efforts
  - But: **verification** of solutions is easy (`one-way' hardness)
    - Discrete log: exponentiation
- **Problem 1: Factoring**
  - Choose randomly $p,q \in_R LargePrimes$
  - Given $n = pq$, it is infeasible to find p,q
  - Verification? Easy, just multiply factors
  - Basis for the RSA cryptosystem and many other tools
- **Problem 2: Discrete logarithm in cyclic group Z_p^***
  - Where p is a safe prime [details in textbook]
  - Given random number, find its (discrete) logarithm
  - Verification is efficient by exponentiation: $O((lg\ n)^3)$
  - Basis for the Diffie-Hellman Key Exchange and many other tools
  - We first discuss key-Exchange problem, then [DH] and disc-log

Key Exchange

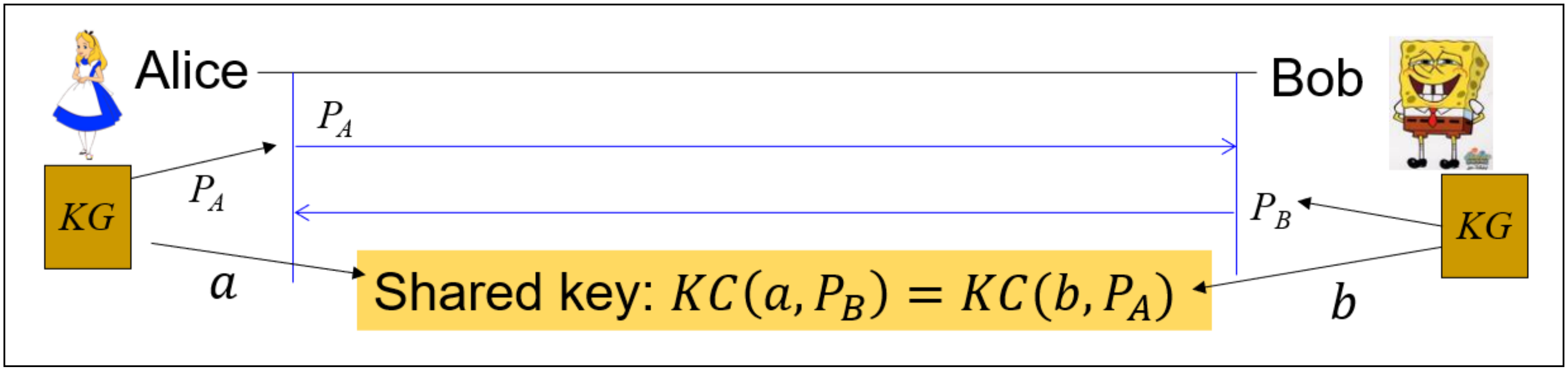
The Key Exchange Problem

Aka key agreement

- Alice and Bob want to agree on secret (key)
  - Secure against **eavesdropper** adversary
  - Assume no prior shared secrets (key)
    - Otherwise seems trivial
    - Actually, we'll later show it's also useful in this case…



Defining a Key Exchange Protocol



Must satisfy correctness; both parties compute the same shared key, and key indistinguishability (the key that the two parties establish is indistinguishable from random).

Discrete Log (DL) Assumption
and
The Computational/Decisional Diffie-Hellman Assumptions (CDH/DDH)
and
The DH Key Exchange Protocol

The Discrete Log Problem

- Computing logarithm is quite efficient over the reals
- Consider a cyclic multiplicative group G
  - Cyclic group: exists <u>generator</u> g s.t. $(\forall a \in G)(\exists i)\left(a = g^i\right)$
  - Discrete log problem: given generator g and $a \in G$, find i such that $a = g^i$
  - Verification: exponentiation (efficient algorithm)
  - For prime p , the group $\mathbb{Z}_p^*$={1,…p-1} is cyclic
- Is discrete-log hard?
  - Some 'weak' groups, i.e., where discrete log is **not** hard:
    - $\mathbb{Z}_p^*$ for prime p, where $(p-1)$ has only 'small' prime factors
      - Using the Pohlig-Hellman algorithm
    - Check!! Mistakes/trapdoors found, e.g., in OpenSSL'16
  - Other groups studied, considered Ok ('hard')
  - **Safe-prime** groups: $\mathbb{Z}_p^*$for **safe prime: $p = 2q + 1$ for prime q**

Discrete Log Assumption
[for safe prime group: $p = 2q + 1$ for prime q]

Given PPT adversary A, and n-bit safe prime p:

$$\Pr\left[\begin{array}{c} g \leftarrow Generator(Z_p^*); \\ x \overset{\$}{\leftarrow} Z_p^* \\ A(x) = a \,|\, x = g^a mod\ p \end{array}\right] \approx negl(n)$$

Comments:

1. Similar assumptions for (some) other groups
2. Knowing q, it is easy to find a generator g
3. Any generator (primitive element) will do

Diffie-Hellman [DH] Key Exchange

Using cyclic group $\mathbb{Z}_p^*$

- Simplified Discrete Exponentiation Key Exchange
- Agree on a random safe prime *p*
  - *And* generator *g* for the cyclic group $\mathbb{Z}_p^*$
- Alice: secret key a, public key $P_A = g^a \bmod p$
- Bob: secret key b, public key $P_B = g^b \bmod p$
- To set up a shared key *:*

Alice

Select a

Bob

Select b

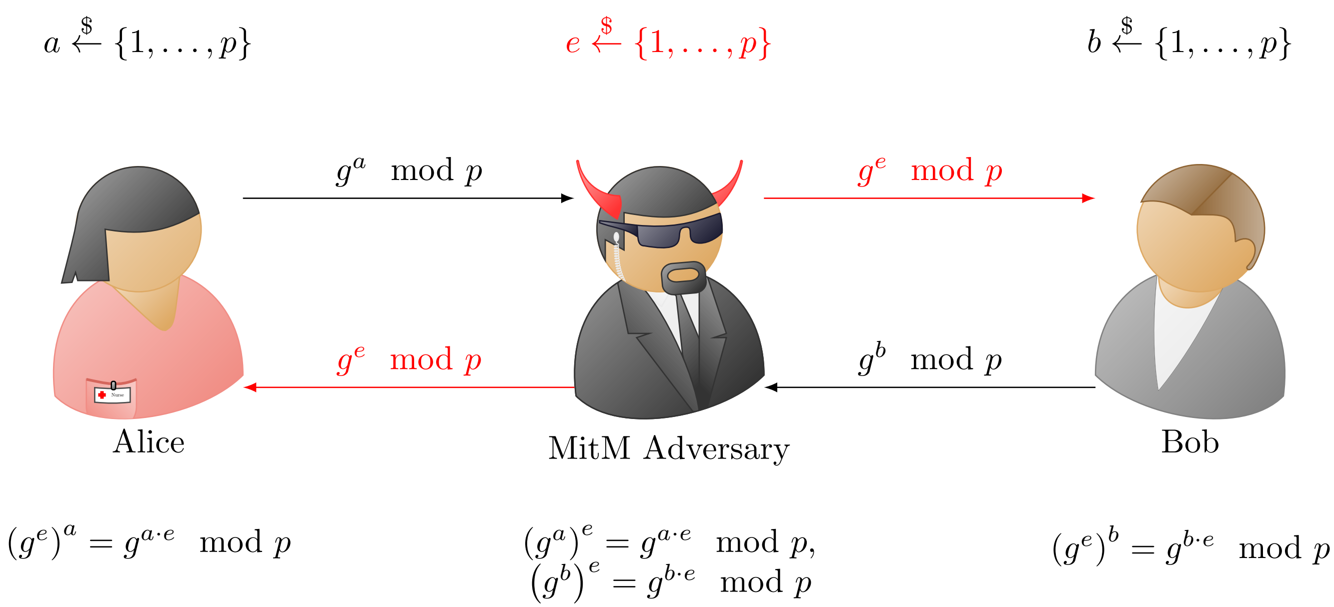
Alice → Bob: $P_A = g^a \bmod p$

Bob → Alice: $P_B = g^b \bmod p$

$$(g^b)^a = g^{ab} = (g^a)^b \bmod p$$

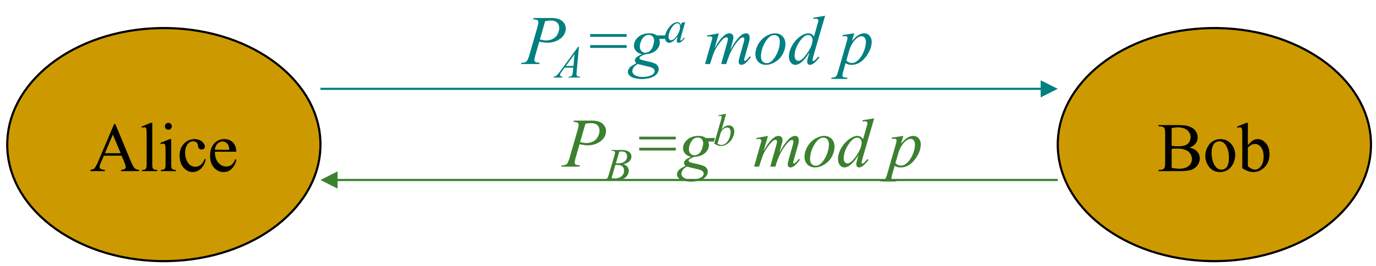
Caution: Authenticate Public Keys!

- Diffie-Hellman key exchange is only secure using the authentic public keys
  - Or (equivalently): against eavesdropper
- If Bob simply receives Alice's public key, [DH] is vulnerable to `Man in the Middle` attack



Security of [DH] Key Exchange

- Assume authenticated communication
- Based on Computational Discrete Log Assumption
- But DH requires stronger assumption than Discrete Log:
  - Maybe from $g^b \bmod p$ and $g^a \bmod p$, adversary can compute $g^{ab} \bmod p$ (without knowing/learning a,b or ab)?



Computational DH (CDH) Assumption [for safe prime group]

Given PPT adversary A:

$$\Pr\left[\begin{array}{c}(p,q) \leftarrow primes\ s.t.\ p = 2q+1; \\ g \leftarrow Generator(Z_p^*); \\ a,b \leftarrow \{1 \ldots p-1\}; \\ A\left(g^a, g^b \bmod p\right) = g^{ab} \bmod p\end{array}\right] \approx negl(n)$$

Assume CDH holds. Can we use g^{ab} as key?

Not necessarily; maybe finding some bits of g^{ab} is easy?

Using DH securely?

- Consider $\mathbb{Z}_p^*$(multiplicative group for (safe) prime p)
- Can g^a , g^b expose *something* about $g^{ab} \; mod \; p$?
- Bad news:
  - Finding (at least) one bit about $g^{ab} \; mod \; p$ is easy!
  - (details in textbook if interested)
- So…how to use DH ‘securely’?

Using DH securely?

- Two options!
  - Option 1: Use DH but with a `stronger' group, e.g., Schnorr's - **not** $\mathbb{Z}_p^*$ (mod safe-prime p)
    - The (stronger) **Decisional DH (DDH) Assumption:** adversary can't **distinguish** between $[g^a, g^b, g^{ab}]$ and $[g^a, g^b, g^c]$, for random a, b, c.
  - Option 2: use DH with safe prime *p… (where only CDH holds)* but use a **key derivation function (KDF)** to derive a secure shared key
  - Applied crypto mostly uses KDF… and we too ☺

Using DH 'securely': CDH+KDF

- **Key Derivation Function (KDF)**
  - Two variants: random-keyed and unkeyed (deterministic)
- Randomized - KDF: $k = KDF_s(g^{ab} mod\ p)$ where KDF is a key derivation function and s is public random ('salt')
- Deterministic - crypto-hash: $k = h(g^{ab} mod\ p)$ where h is randomness-extracting crypto-hash
  - No need in salt, but **not** provably-secure

Resilience to Key Exposure

Authenticated DH

- Recall: DH is not secure against MitM attacker
- Use DH for **resiliency to key exposure**
  - Do authenticated DH periodically
  - Use derived key for confidentiality, authentication
    - Some protocols use key to authenticate next exchange
  - ➔ Perfect Forward Secrecy (PFS):
    - Confidentiality of session i is resilient to exposure of all keys, except i-th session key, after session i ended

Authenticated DH: using KDF/PRF [TLS]

- Assume f which is <u>both</u> a PRF and a KDF
- MK is secret + f is PRF (& MAC) ➔ authentication
  - And, assuming MK is secret, session keys are secure – even if discrete-log would be easy (quantum computers or math breakthrough)
- Assuming CDH and that f is **KDF**: secure if MK exposed
  - Since most bits of $g^{a_i b_i}$ are secret
  - Against eavesdropping or if MK is exposed only after session ends.
  - Perfect forward secrecy (PFS) !

Alice

Bob

Alice → Bob: $g^{a_i} \, mod \, p, f_{MK}(g^{a_i} \, mod \, p)$

Bob → Alice: $g^{b_i} \, mod \, p, f_{MK}(g^{b_i} \, mod \, p)$

Session key: $k_i = f_{MK}\left(g^{a_i b_i} \, mod \, p\right)$

Alice → Bob: $MAC_{k_i}(A, B, m)$

Resilience to Key Exposure: Recover Security

- The previous DH protocol does not achieve recover security, why?
  - Exposing ML makes all future session vulnerable to MitM (this adversary can authenticate any public key he wants to the other party).
- There is another version, called Ratchet DH, that achieves perfect recover security.
  - Will not be covered in this class.

Covered Material From the Textbook

- Appendix A.2
- Chapter 6: sections 6.1, 6.2, and 6.3 (except 6.3.2)

Thank You!