CSE 3400/ CSE 5850 - Introduction to Cryptography & Cybersecurity
/ Introduction to Cybersecurity

Lecture 6

Hash Functions – Part I

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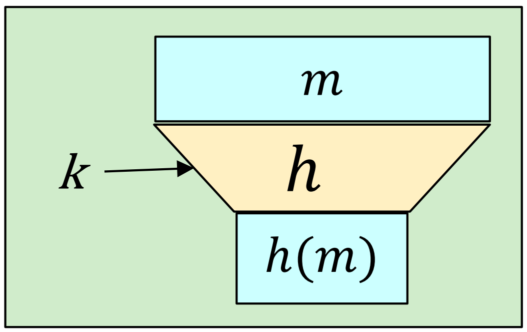
Adapted from textbook slides

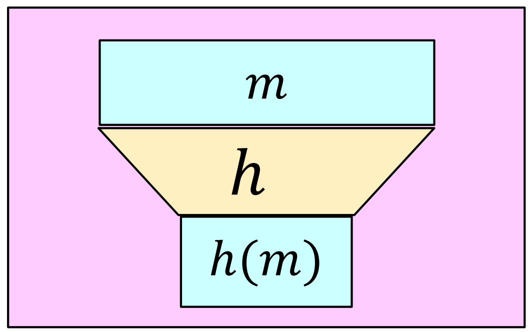
Outline

- Introduction and motivation.
- Collision resistant hash functions (CRHF).
- CRHF applications.
- Other notions of security.
  - TCR, SPR, OWF.
- Randomness extraction.
- The random oracle model.

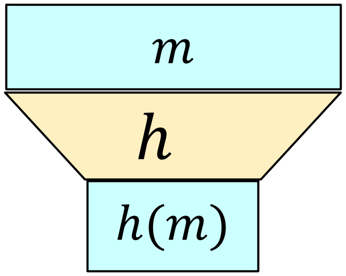
Hash Functions

- Input m: binary strings
- Output $h(m)$:
  - 'Short' (n-bit) binary strings
    - Aka **message digest**
- Efficiently computable
- Applications: cryptography, security, efficiency
- Keyed $h_k(m)$, where the key is public, or unkeyed $h(m)$





Hash functions: simple examples



- For simplicity: input m is decimal integer
  - View as string of (three) digits
  - For example, $m = 127 \rightarrow m_1 = 1, m_2 = 2, m_3 = 7$
- Least Significant Digit hash:

$$h_{LSD}(m) = m_3$$

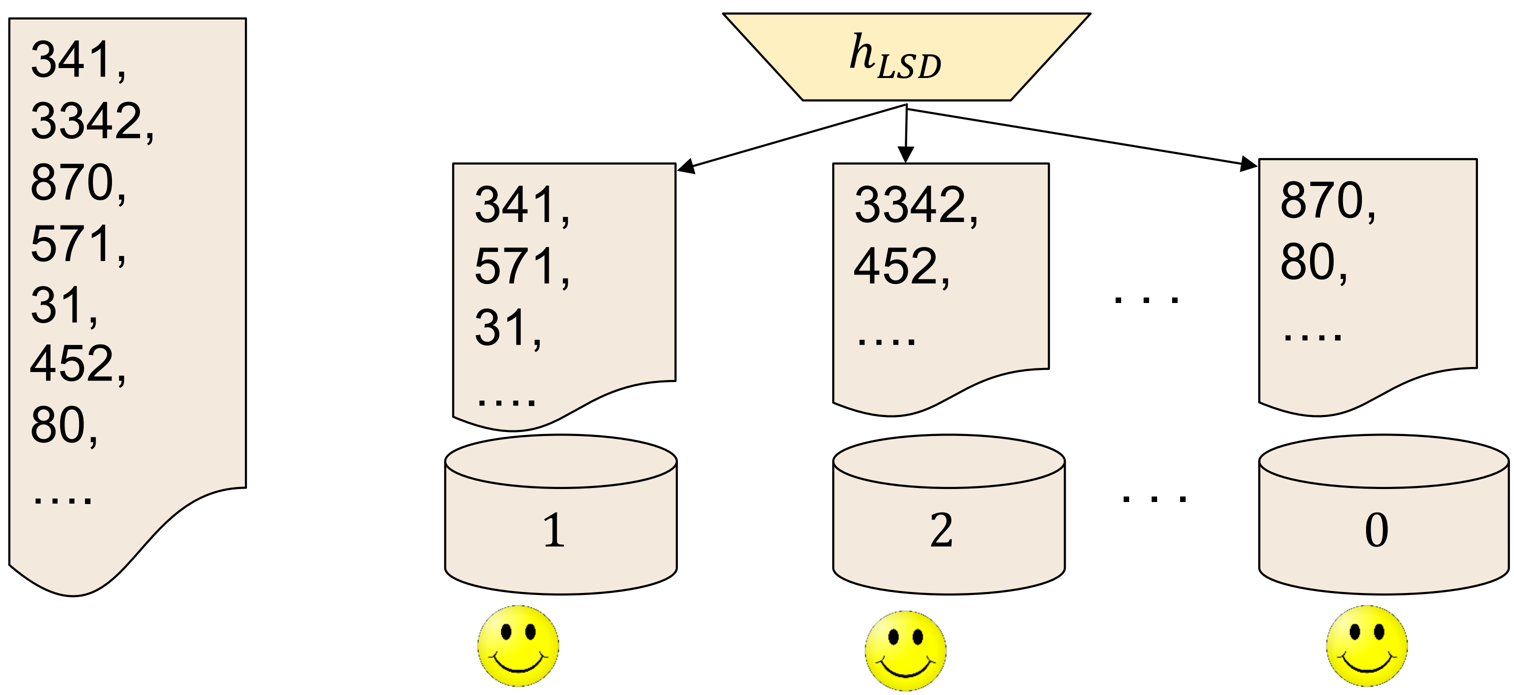
- Sum hash: $h_{Sum}(m) = (m_1 + m_2 + m_3) \; mod \; 10$
- Exercise:

$$h_{LSD}(117) = \underline{7}$$
$$h_{Sum}(117) = \underline{9}$$

> Note: the above are insecure hash functions, these are just toy examples to grasp the concept of hashing.

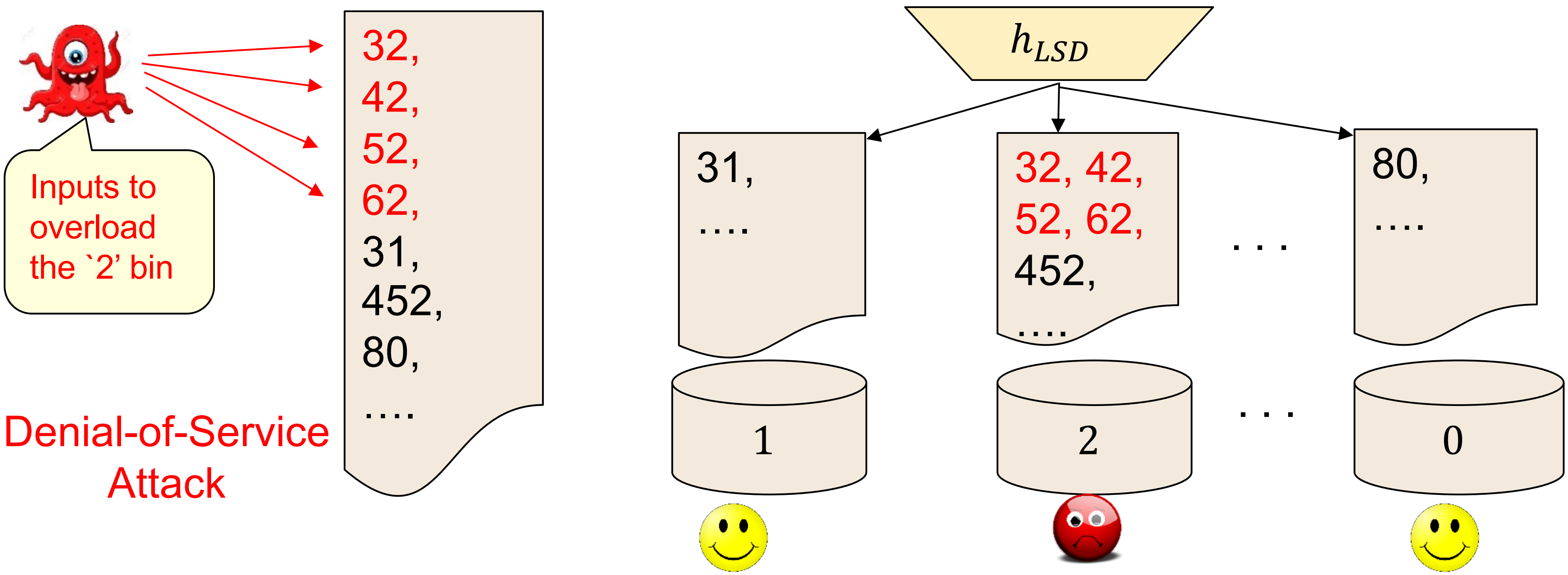
Motivation: Hashing for efficiency

- Input: large set (e.g., integers or strings)
- Goal: map `randomly' to few bins
  - E.g., to ensure efficiency – load balancing, etc.



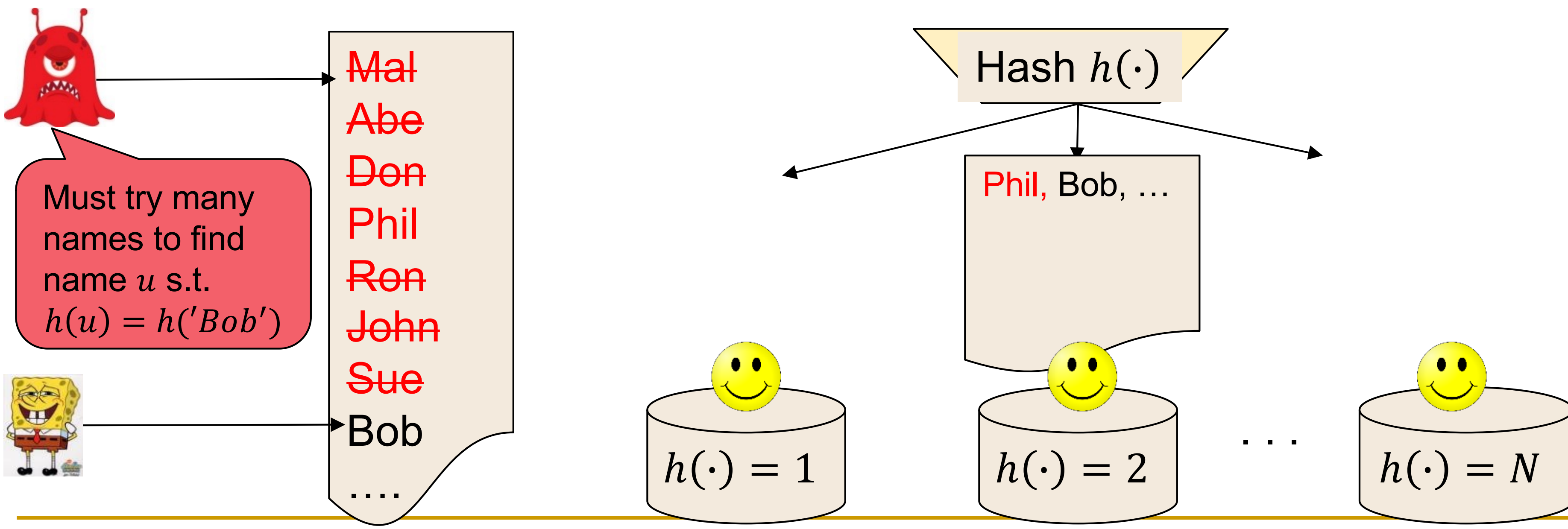
Collisions?

- Input: large set (e.g., integers or strings)
- Goal: map `randomly' to few bins
  - E.g., to ensure efficiency – load balancing, etc.
  - Adversary chooses inputs that hash to same bin



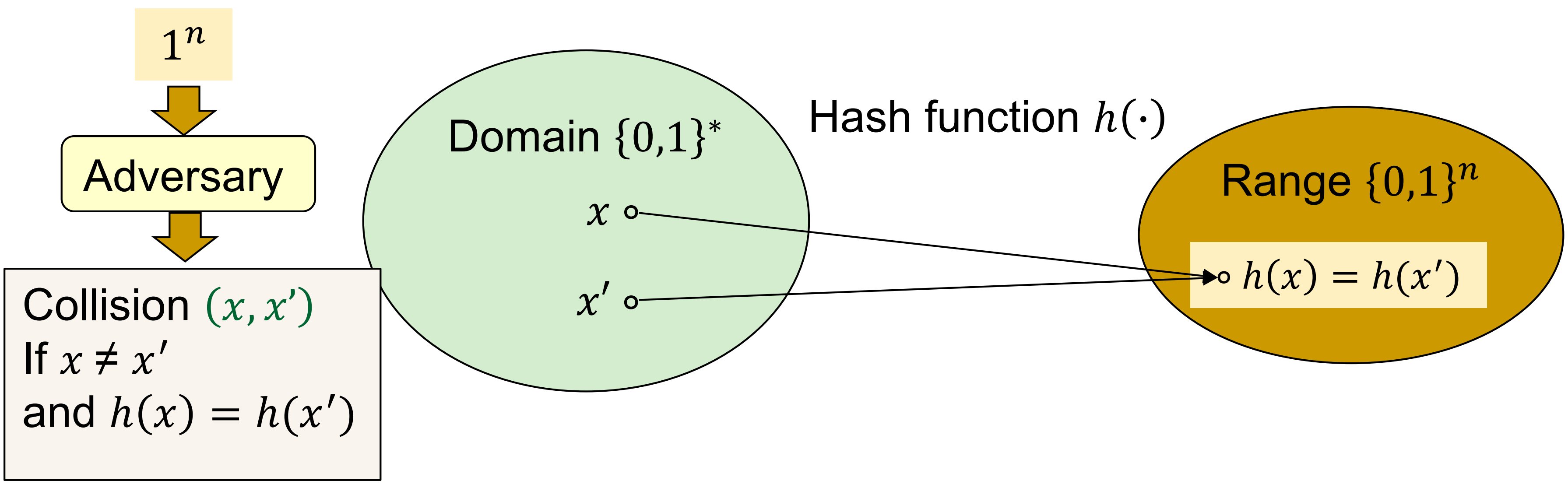
Security Goal: Collision Resistance

- A **collision:** two inputs (names) with same hash: $h('Bob')=h('Phil')$
- Every hash has collisions, since |input|>>|output| !
- Collision resistance: hard to **find** these collisions
  - Note: attacker can always try names randomly until a collision is found
  - But this should be ineffective: must try about (on average) N names (number of bins)



Collision Resistant Hash Function (CRHF)

- h is CRHF if it is hard to **find** collisions $h(x)=h(x')$ for $x \neq x'$
  - Note: attacker can always try inputs randomly till finding collisions
  - But this should be ineffective: must try about $|Range|$ values
- Hard means that the probability that the attacker succeeds in finding a collision is negligible.



Collision Resistant Hash Function (CRHF)

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Definition . (Keyless Collision Resistant Hash Function (CRHF)). *A keyless hash function* $h^{(n)}(\cdot) : \{0,1\}^* \rightarrow \{0,1\}^n$ *is* collision-resistant *if for every efficient (PPT) algorithm* $\mathcal{A}$, *the* advantage $\varepsilon^{CRHF}_{h,\mathcal{A}}(n)$ *is* negligible *in* n, *i.e., smaller than any positive polynomial for sufficiently large* n *(as* $n \rightarrow \infty$*), where:*

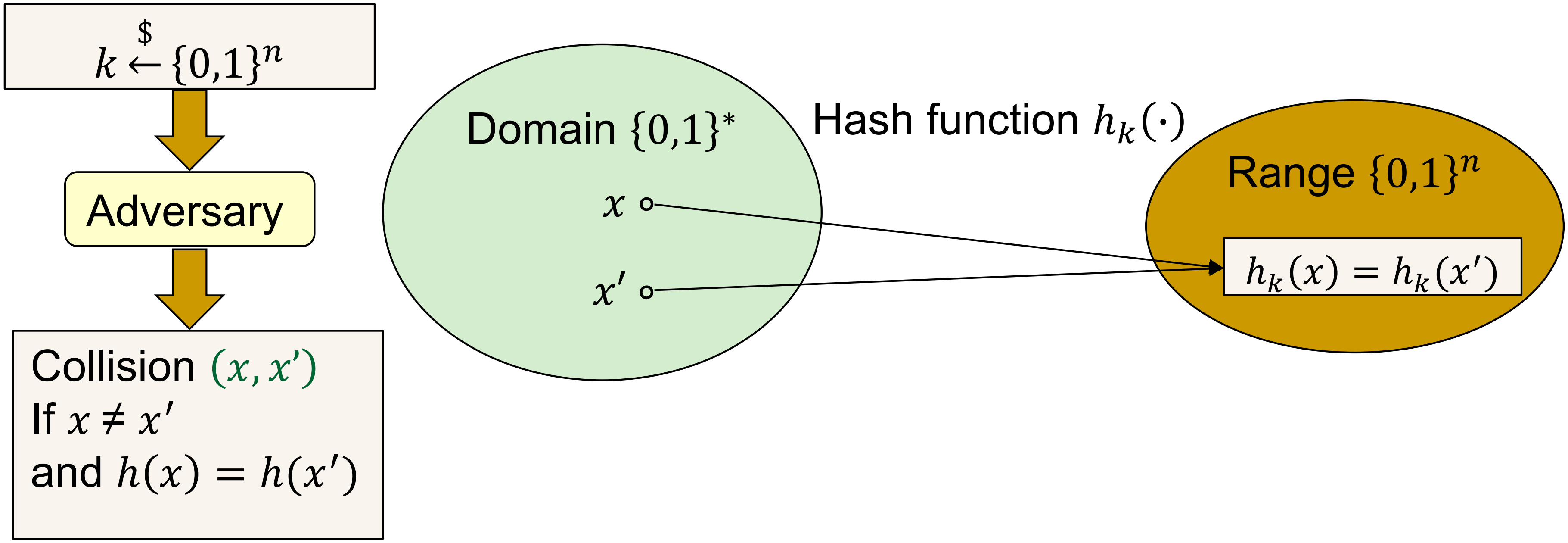
$$\varepsilon^{CRHF}_{h,\mathcal{A}}(n) \equiv \Pr\left[(x,x') \leftarrow \mathcal{A}(1^n) \ s.t. \ (x \neq x') \wedge (h^{(n)}(x) = h^{(n)}(x')\right]$$

Where the probability is taken over the random coin tosses of $\mathcal{A}$.

Keyless CRHF **Do Not** Exist!

- |Range|<<|Domain| so there is a collision where $h(x')=h(x)$, $x \neq x'$
- For a keyless CRHF there is a PPT algorithm A that can always output a collision: $A(1^n) = \{return\ x, x'\}$
  - Proof: in textbook.
    - Intuitively, since the function is fixed (same input-output mapping), a collision instance can be hardcoded in the attacker algorithm and just output that collision and win the security game.
- Solutions:
  - keyed CRHF,
  - use weaker collision notions,
  - or ignore! (more like asking if the collision is useful for the attacker?)

Keyed CRHF



Adversary knows k but **not in advance** – cannot `know` a collision

Often referred to as **ACR**-hash (**ANY**-collision resistance)

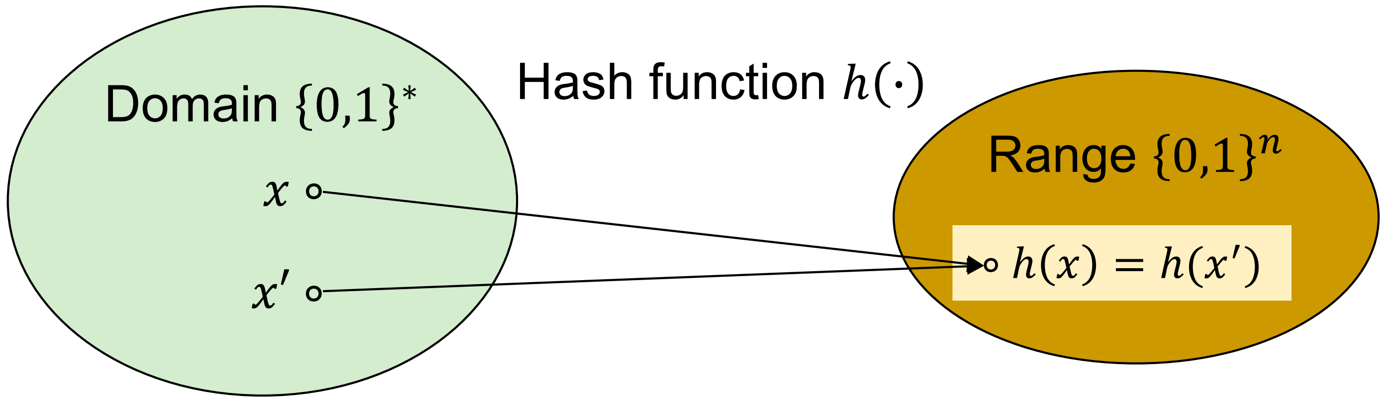
Keyed CRHF - Definition

Definition (Keyed Collision Resistant Hash Function (CRHF)). *Consider a keyed hash function* $h_k(\cdot) : \{0,1\}^n \times \{0,1\}^* \to \{0,1\}^n$*, defined for any* $n \in \mathbb{N}$*. We say that* h *is* collision-resistant *if for every efficient (PPT) algorithm* $\mathcal{A}$*, the* advantage $\varepsilon^{CRHF}_{h,\mathcal{A}}(n)$ *is* negligible *in* n*, i.e.,* $\varepsilon^{CRHF}_{h,\mathcal{A}}(n) \in NEGL(n)$*, where:*

$$\varepsilon^{CRHF}_{h,\mathcal{A}}(n) \equiv \Pr_{k \leftarrow \{0,1\}^n} [(x, x') \leftarrow \mathcal{A}(k) \ s.t. \ (x \neq x') \wedge ((h_k(x) = h_k(x'))]$$

Where the probability is taken over the random coin tosses of the adversary $\mathcal{A}$ *and the random choice of* k*.*

Generic Collision Attacks



- An attacker that runs in exponential time can always find a collision (i.e., non PPT attacker)
  - Easy: find collisions in 2^n time by trying 2^n + 1 distinct inputs (compute their hash and locate a collision).
- An attacker finds a collision with 2^{-n} probability (negligible probability).
  - Choose *x* and *x'* at random and check if they produce a collision.

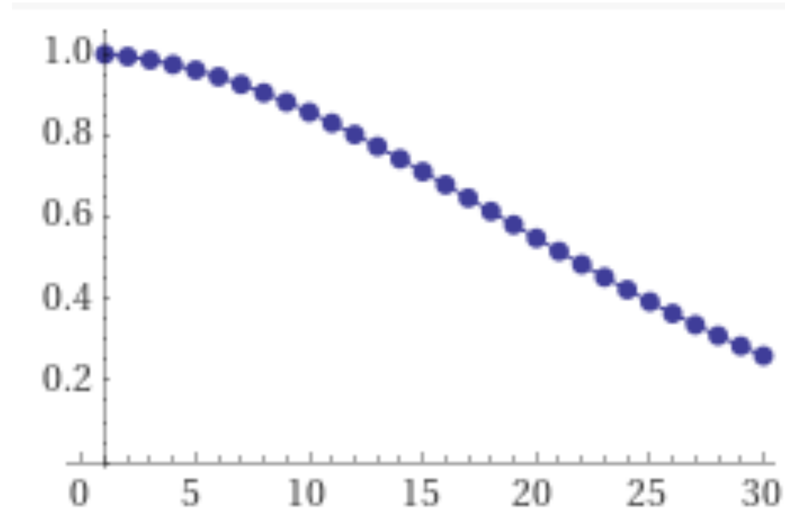
The Birthday Paradox

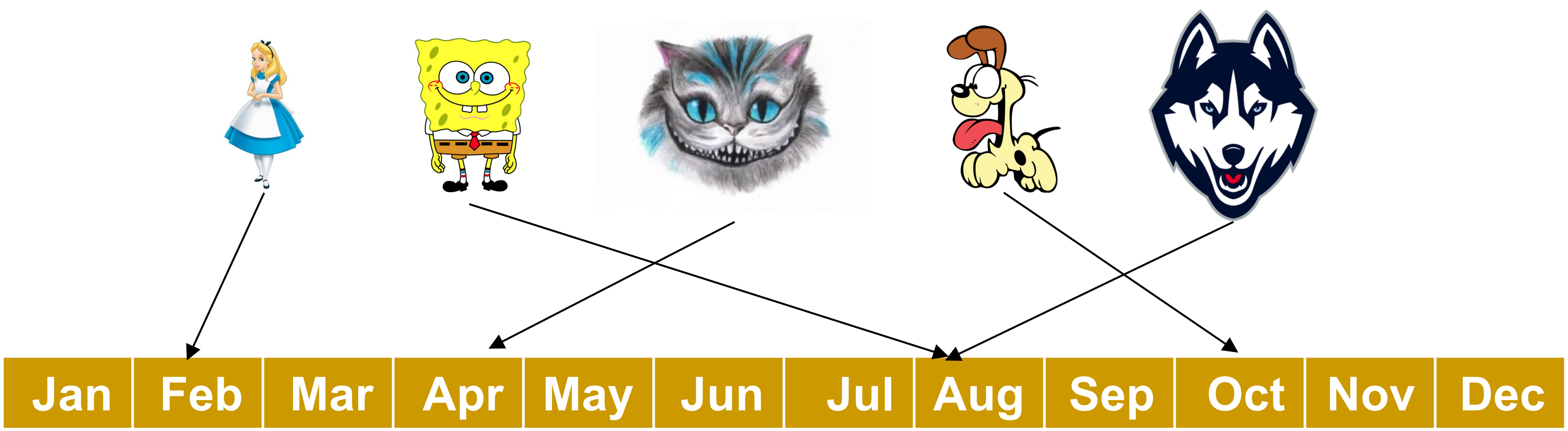
- **The birthday paradox** states that the expected number q of hashes (i.e., hash queries) until a collision is found in $O(2^{n/2})$ not $O(2^n)$.
  - It is $q \lessapprox 2^{n/2} \cdot \sqrt{\frac{\pi}{2}} \lessapprox 1.254 \cdot 2^{n/2}$
- For 80 bit of effective security, use n=160 !
  - So to defend against an attacker who can perform 2^{80} hashes set the digest length to be at least 160 bits.
    - So the range has a size of 2^{160} digests.
- Why? Intuition?

The Birthday Attack ('Paradox')

- Probability of NO birthday-collision:
  - Two persons: (364/365)
  - Three persons: (364/365)*(363/365)
  - …
  - n persons: $\prod_{i=1}^{n-1} \frac{365-i}{365}$
  - So for 23 individuals, the probability

  of birthday collision is about 0.5



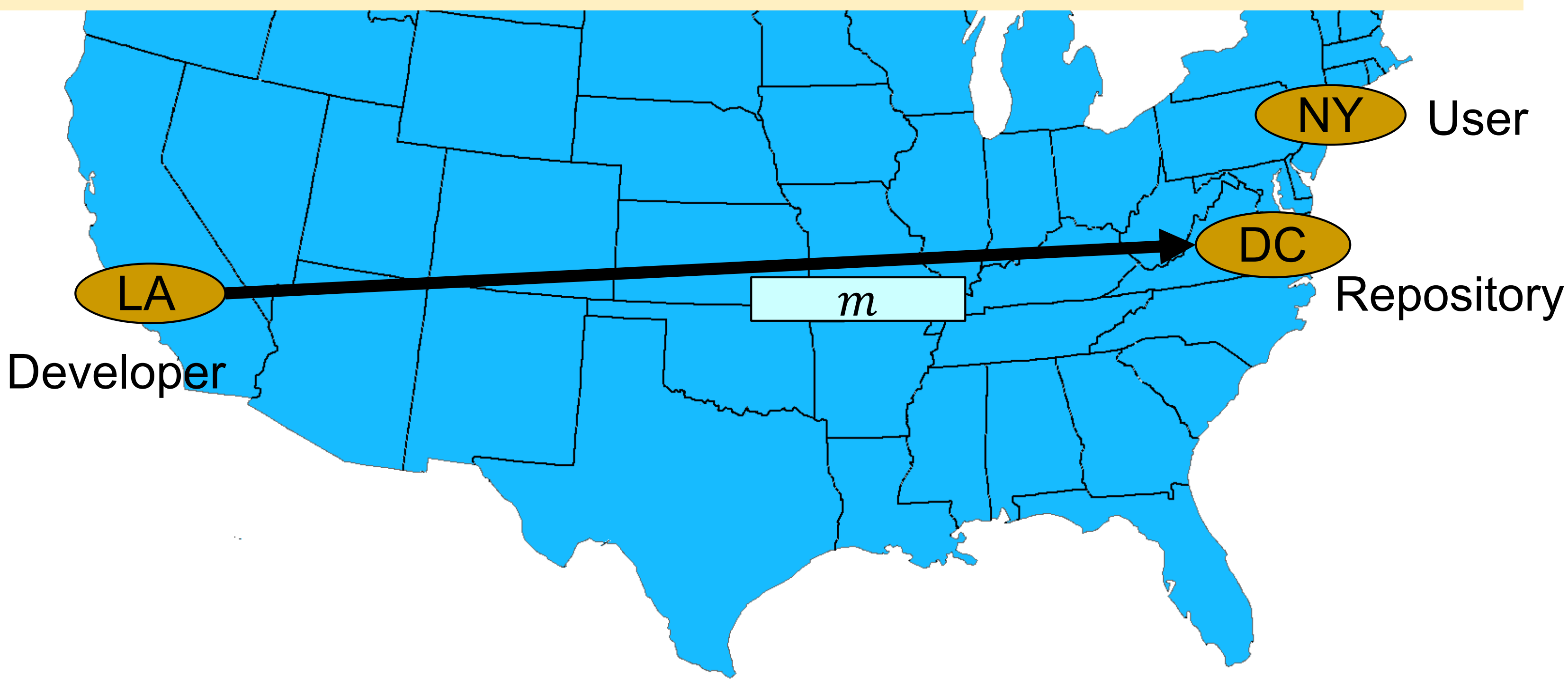


Collision-Resistance: Applications

- Integrity (of object / file / message)
  - Send *hash(m)* securely to validate *m*
  - Later we will see how a hash function can be used to construct a MAC (called HMAC).
- Hash-then-Sign
  - Instead of signing *m* sign *hash(m)*
    - More efficient!
    - We will explore this in detail once we study digital signatures.
- Blockchains
  - Later

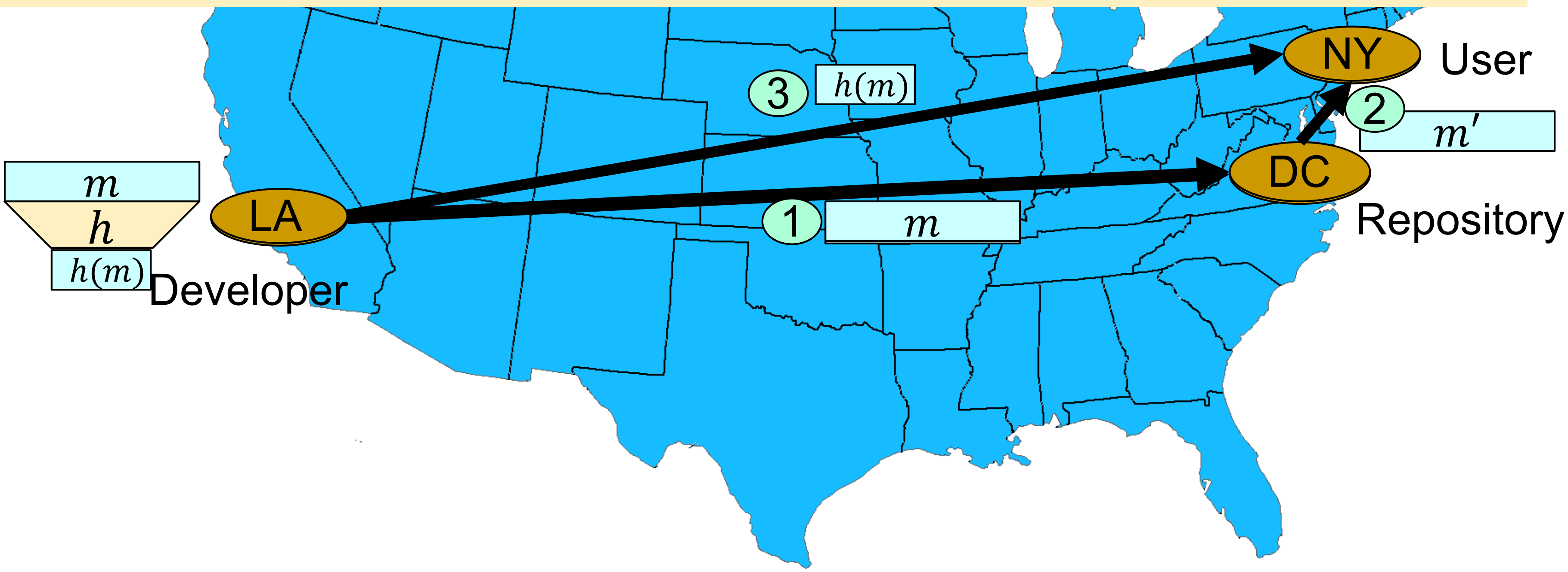
CRHF and Software Distribution

- Developer in LA develops a large software m
- Repository in DC obtains a copy of m
- User in NY wants to obtain m – securely and efficiently
  - Don't send m from LA to **both** NY and DC
- How?



CRHF: secure, efficient SW distribution

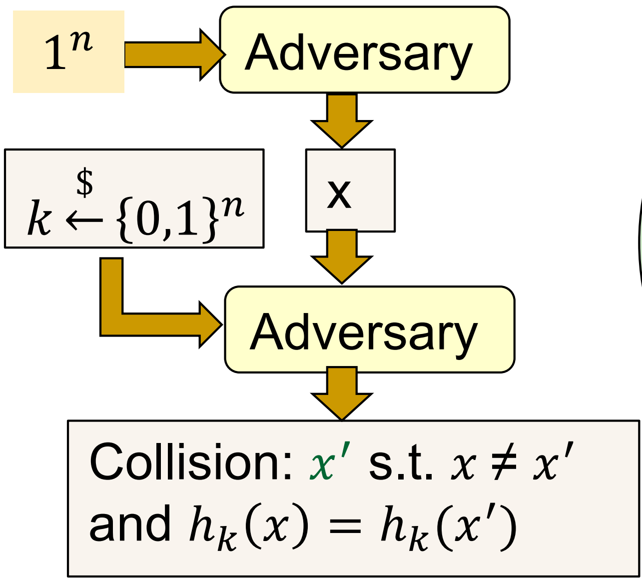
1. Repository in DC downloads software m from developer in LA
2. User downloads from (nearby) repository; receives m'
   - Is $m' = m$? User should validate! How?
3. User securely downloads $h(m)$ directly from developer
   - Digest $h(m)$ is short – much less overhead than downloading m
4. User validates: $h(m) = h(m') \rightarrow m = m'$

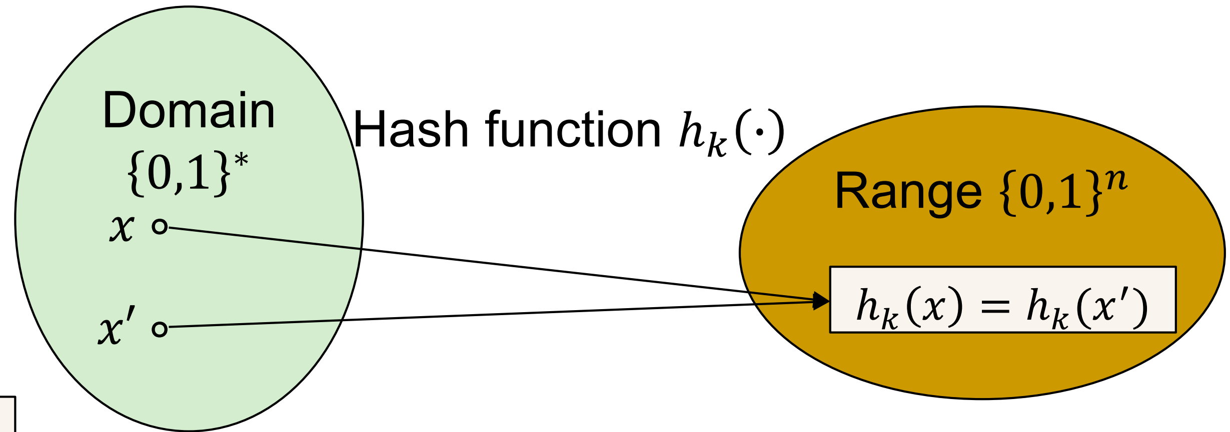


Other Notions of Security

- Collision resistance provides the strongest guarantee.
  - Gives more freedom to the adversary; the adversary wins if it finds any two inputs with the same digest.
    - No conditions on these two inputs other than being in the domain of the hash function.
- Other security notions (but sufficient for many applications):
  - Target collision resistance (TCR).
  - Second preimage resistance.
  - First preimage resistance.
- Birthday paradox (or attack) does not work against these weaker notions.
  - It is for collision resistance; find ***any*** two inputs that collide!

Target CRHF (TCR Hash Function)

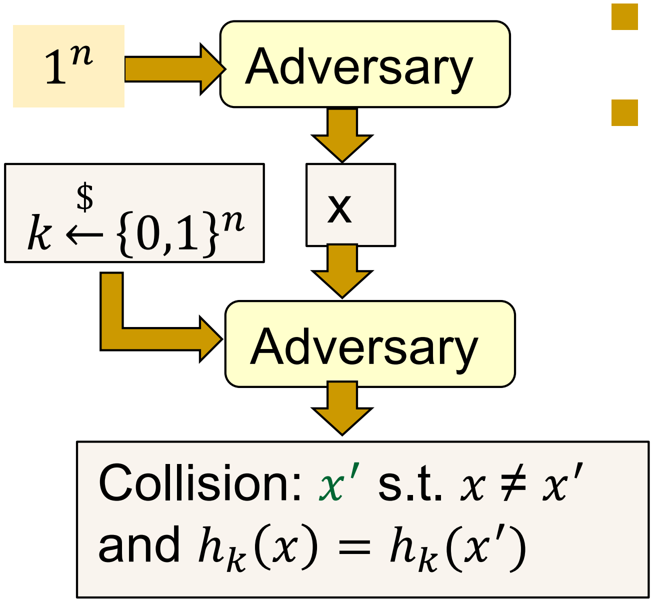




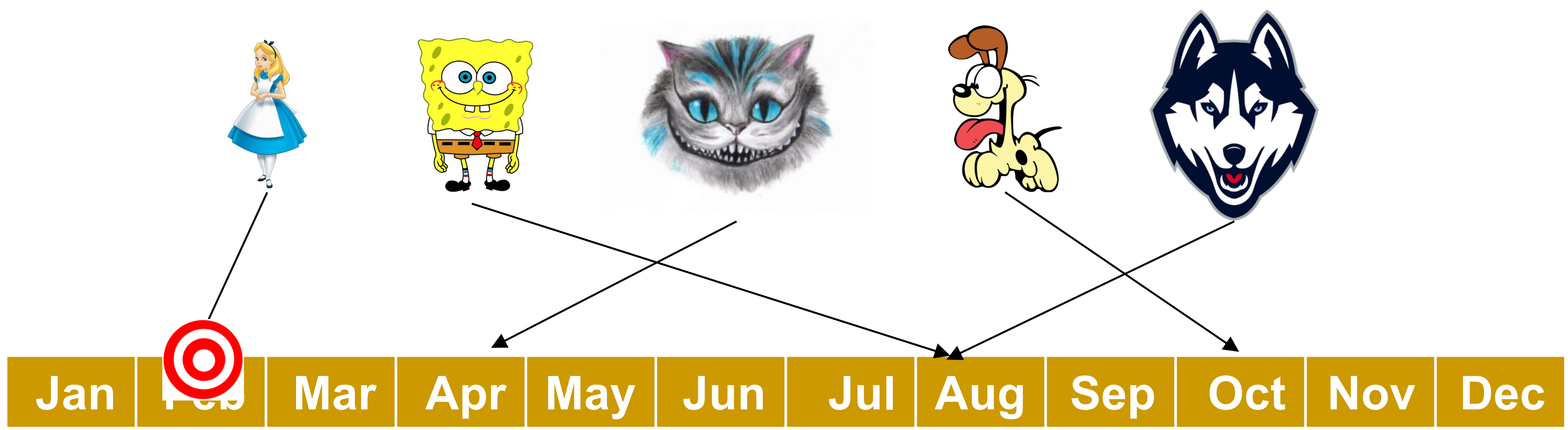
Adversary has to select target **before** knowing key

$$\varepsilon_{h,\mathcal{A}}^{TCR}(n) \equiv \Pr_{k \leftarrow \{0,1\}^n}\left[\left\{\begin{array}{c} x \leftarrow A(1^n); \\ x' \leftarrow A(x,k) \end{array}\right\} \; s.t. \;\; (x \neq x') \wedge (h_k(x) = h_k(x')\right]$$

TCR and Birthday Paradox?



- **First:** adversary selects **x**
- Probability for NO birthday-collision **with x:**
  - Two persons: (364/365)
  - Three persons: (364/365)*(36**4**/365)
  - …
  - n persons: $\prod_{i=1}^{n-1} \frac{364}{365} = \left(\frac{364}{365}\right)^{n-1}$

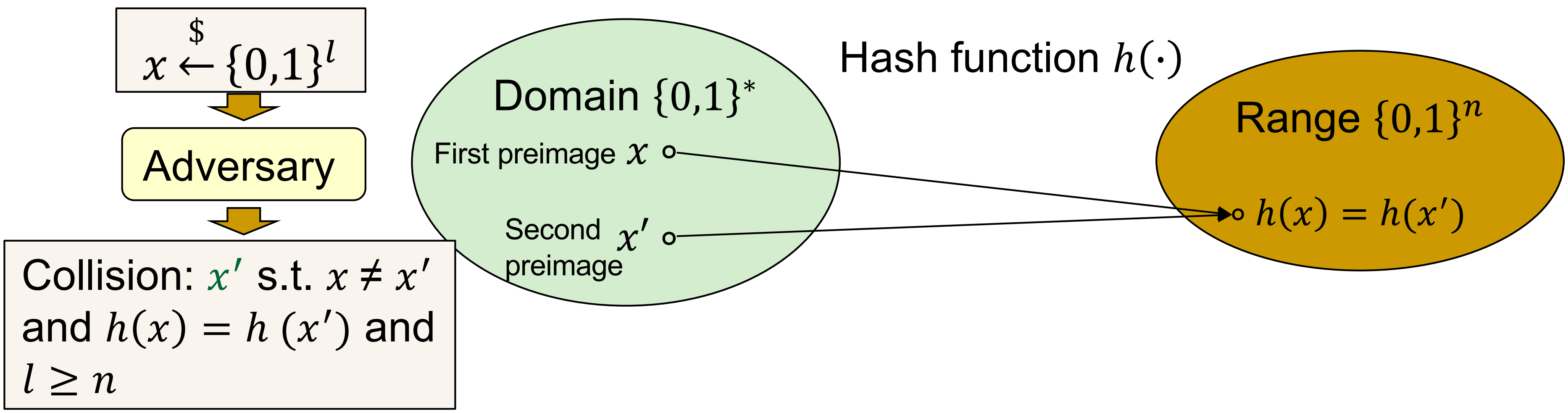


We (mostly) focus on keyless hash…

- Although there are no CRHFs and theory papers focus on keyed hash
- But…
  - It's a bit less complicated and easier to work with.
  - No need to consider both ACR (this stands for Any Collision Resistance, which is the same as CRHF) and TCR
    - Why?
  - Modifying to CRHF is quite trivial
    - Just make it keyed!
  - Usually used in practice: code libraries, standards, …

2^{nd}-Preimage-Resistant Hash (SPR)

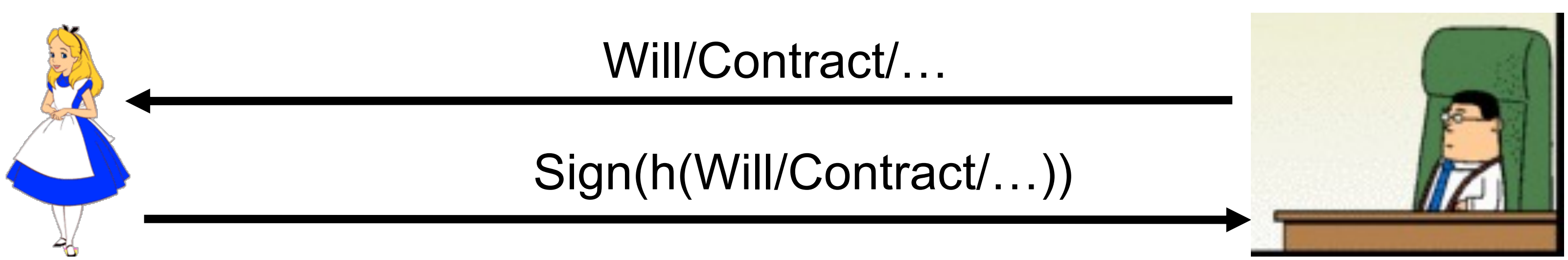
- Hard to find collision with a <u>specific random x</u>.
  - So the attacker's advantage, probability of finding such a collision, in the following game is negligible.



Use with care!
(think carefully about the security you want to achieve and see if SPR suffices)

CRHF/SPR vs. Applications

- CRHF secure for signing, SW-distribution
- How about SPR hash (weak-CRHF)?
  - SW-distribution? ***YES***
  - Hash-then-sign? ***NO***
- Why?
  - Attacker can't impact SW to be distributed
  - But… attacker may be able to impact signed msg!

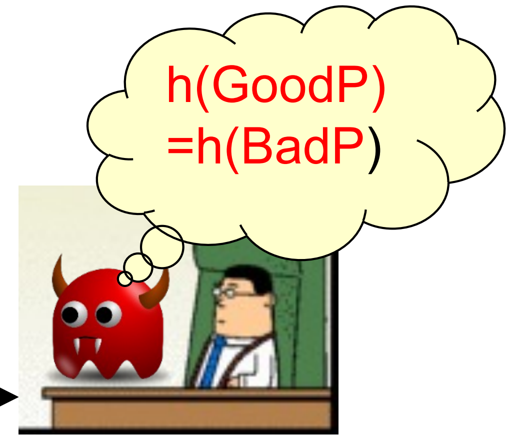


SPR: Collisions to **Chosen** Messages

- Or: Alice and Mal, the corrupt lawyer
- Mal finds two `colliding wills', GoodW and BadW:
  - GoodW: contents agreeable to Alice
  - h(GoodW)=h(BadW)
  - Alice Signs good will: Sign(h(GoodW))

GoodW: 'I leave all to Bob'

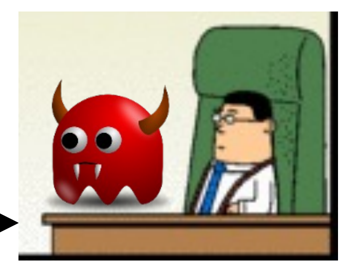
Sign$_A$(h(GoodW))



- Later… Mal presents to the court:

BadW: 'I leave all to Mal', Sign$_A$(h(BadW))

$$$$



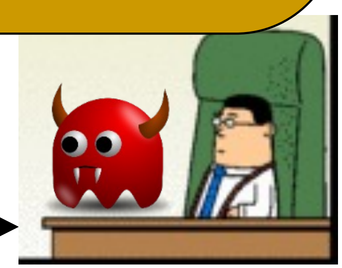
SPR: collisions to **chosen** message

- Or: Alice and Mal, the corrupt lawyer
- Mal finds two `colliding wills', GoodW and BadW:
  - GoodW: contents agreeable to Alice

Is such attack realistic?
Or SPR is enough 'in practice'? Yes,
usually rely on having a specific
format for messages, like a fixed
prefix chosen by the attacker.

BadW: 'I leave all to Mal', $Sign_A$(h(BadW))

$$$$



Examples

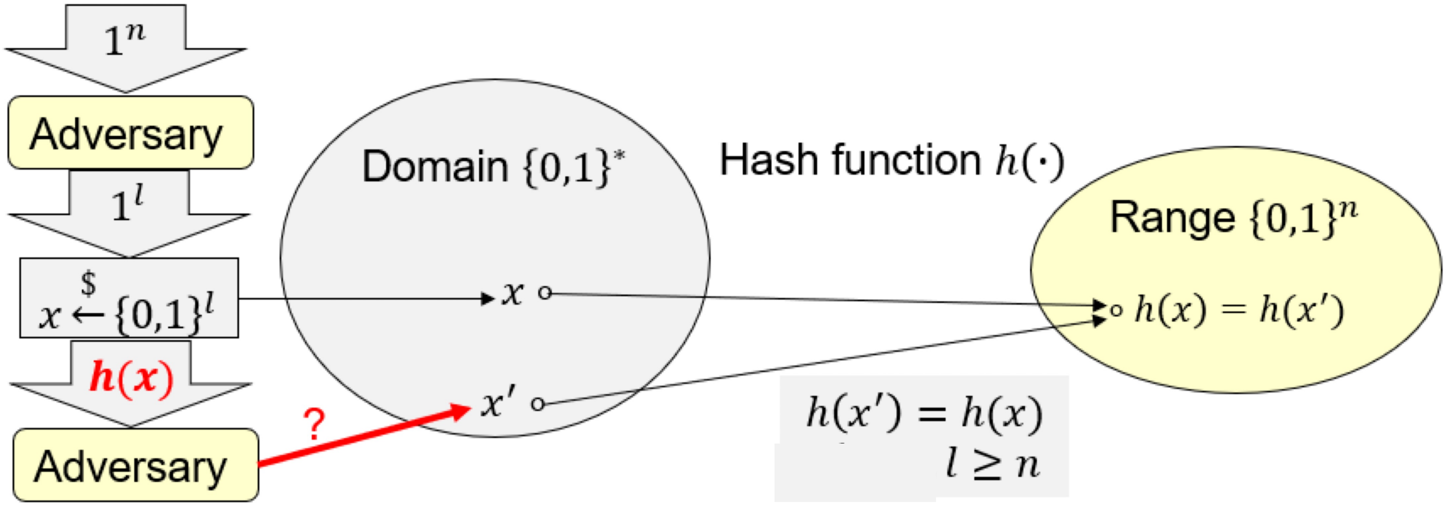
1) Let h_k be a keyed CRHF. Is $h_k' = h_k(h_k(x))$ a CRHF? Why?

2) For x parsed as x = $x_1||x_2||x_3$, let $h(x_1||x_2||x_3) = x_1 + x_2 + x_3 \ mod \ p$, is h a CRHF? Why? Is it an SPR? Why?

3) Let $h_k(m)$ be a TCR function. Construct $h_k'(m) = 0^n$ if $m[1{:}\ |k|] = k$ and $h_k(m)$ otherwise (recall that the hash function input can be of any length).

- Is h_k' *a* CRHF? Why?
- Is h_k' a TCR? Why?

One-Way Function (OWF)

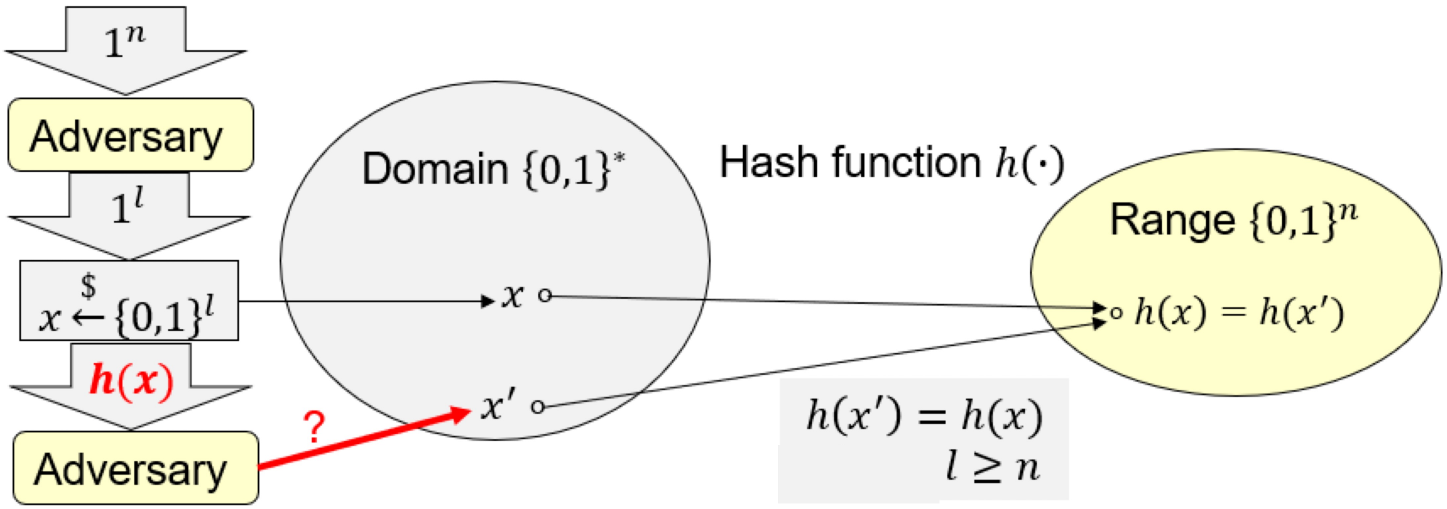


- **One-way function** or **first preimage resistance**: given $h(x)$ for a random x, it is hard to find x, or any x' s.t. $h(x')=h(x)$

Compare to:

- Collision-Resistance (CRHF): hard to find any collision, i.e., any (x, x') s.t. $h(x')=h(x), x \neq x'$
- Second-preimage resistance (SPR): hard to find a collision with random x, i.e., x' s.t. $h(x')=h(x), x \neq x'$

Application: One-time Password Authentication



- **One-time password authentication:**
  - Select random x : 'one-time password' (keep secret!)
  - Validate using non-secret 'one-time validation token': $h(x)$

Randomness Extraction

- It is desired property of hash functions.
- 'If input is sufficiently random, then output is random'
- **Randomness extraction**: if any m input bits are random ➔ all n output bits are pseudorandom
  - For sufficiently large m

Exercise

- Let h_1, h_2 be both CRHF and OWF
- Use them to construct:
  - h_{CRHF} - CRHF but not OWF
  - h_{OWF} - OWF but not CRHF

Exercise

- Let h_1, h_2 be both CRHF and OWF
- Use them to construct:
  - h_{CRHF} - CRHF but not OWF
  - h_{OWF} - OWF but not CRHF
- One possible solution:
  - $h_{CRHF}(m) = \begin{cases} m & if\ |m| = n \\ h_1(m) & otherwise \end{cases}$
  - $h_{OWF}(m) = \begin{cases} h_1(m)\ if\ |m| = n \\ h_1\left(m_{1..n} \oplus h_2(m')\right)\ if\ m = m_{1..n} || m' \end{cases}$

What does all of that tell you about …

- *… the relation between CRHF, TCR, SPR, and OWF?*
  - CRHF vs. TCR?
  - CRHF vs. SPR?
  - TCR vs. SPR?
  - CRHF vs. OWF?
  - SPR vs. OWF?

Covered Material From the Textbook

- Chapter 3
  - Section 3.1,
  - Section 3.2,
  - Section 3.3, (except 3.3.6, and from 3.3.5 only what we covered in class)
  - Section 3.4 (except 3.4.2),
  - Section 3.5 (only its introduction, i.e., text before 3.5.1),

Thank You!